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THE FAMILY OF ALL RECURSIVELY ENUMERABLE CLASSES OF FINITE SETS(1)

BY
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Abstract. We prove that if $P(x)$ is any first-order arithmetical predicate which enumerates the family Fin of all r.e. classes of finite sets, then $P(x)$ must reside in a level of the Kleene hierarchy at least as high as $\Pi_3^0 - \Sigma_3^0$. (It is more easily established that some of the predicates $P(x)$ which enumerate Fin do lie in $\Pi_3^0 - \Sigma_3^0$.)

1. **Introduction.** It has been remarked by C. E. M. Yates, in a footnote on p. 338 of [7], that the family Fin of all recursively enumerable classes of finite subsets of N (N = the set of all natural numbers) is not a recursively enumerable family. Since the family Fin is involved in many of the constructions which occur in recursion theory, the exact location of its "level of enumerability" in the Kleene hierarchy seems to us to be a natural and perhaps even useful undertaking. It is trivial to show that Fin can be enumerated by a Σ_4^0 predicate; thus, since Yates' remark amounts to the assertion that Fin cannot be enumerated by a Σ_1^0 predicate, it remains to consider the levels Σ_2^0 and Σ_3^0. We settle the matter by means of Theorem 2 in §3, which implies: Fin cannot be enumerated by a Σ_3^0 predicate. On the other hand, it follows from a theorem of Rogers [6, p. 326, Theorem XV] that the index set $G(\text{Fin})$, as defined below, is a complete Π_3^0 set of numbers; in §2 we shall prove a general assertion about index sets corresponding to families, from which Rogers' theorem is readily derivable.

In the remainder of this introductory section, we set forth the notational and terminological conventions which are to be in force in §§2, 3. N denotes the set $\{0, 1, 2, \ldots\}$ of natural numbers. Lower case Greek letters other than δ and ρ denote either subsets of N or partial number-theoretic functions (i.e., functions from a domain β into N, where $\beta \subseteq N \times \cdots \times N$ (k times) for some $k \geqq 1$); context will determine, in any particular case, whether a subset or a function is meant. For each $k \geqq 1$, let the predicate $T_k(z, x_1, \ldots, x_k, y)$ be defined as in [4, §57]. Then with

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notation as in [4, §63], we obtain a recursive enumeration $\{\varphi_e^k\}_{e=0}^{\infty}$ of the k-place partial recursive functions by means of the definition:

$$\varphi_e^k(x_1, \ldots, x_k) \simeq U(\mu y T_k(e, x_1, \ldots, x_k, y)).$$

It is easily seen from the definition of T_k that $\varphi_0^k = \varnothing$ for all k. If ψ is a partial number-theoretic function, we denote by $\delta\psi$ the domain of ψ and by $\rho\psi$ the range of ψ. If ψ is a partial number-theoretic function of k variables, where $k \geqq 2$, and if $a_1, \ldots, a_{k-1}$ are constants, then by $\delta\psi(a_1, \ldots, a_{k-1}, x)$ we mean

$$\{x \mid \psi(a_1, \ldots, a_{k-1}, x) \textit{ is defined}\}.$$

For each e, we denote by W_e the set $\delta\varphi_e^1$. Thus in our notation W_e = the eth recursively enumerable set; we abbreviate "recursively enumerable" as "r.e.". For each $k \leqq 1$, we denote by Σ_k^0 the class of predicates $P(x)$ such that, for some e, $P(x) \Leftrightarrow (\exists x_1)(\forall x_2) \cdots (Qx_k)T_k^*(e, x, x_1, \ldots, x_k)$, where the quantifiers alternate between existential and universal and T_k^* is either T_k or $\neg T_k$ according as k is odd or even ($Q = \exists$ if k is odd; $Q = \forall$ if k is even). We denote by Π_k^0 the class $\{R(x) \mid$ for some $P(x) \in \Sigma_k^0$ we have $P(x) \Leftrightarrow \neg R(x)\}$; i.e., the predicates in Π_k^0 are the negations of those in Σ_k^0. It is well known that the class Σ_k^0 includes all predicates expressible in the form $(\exists x_1)(\forall x_2) \cdots (Qx_k)R(x, x_1, \ldots, x_k)$ where $R(x, x_1, \ldots, x_k)$ is a recursive predicate. We shall denote by $\Sigma[k; e]$ the set

$$\{x \mid (\exists x_1)(\forall x_2) \cdots (Qx_k)T_k^*(e, x, x_1, \ldots, x_k)\}, \qquad k \geqq 1.$$

(Note that $\Sigma[1; e] = W_e$ for all e.) A predicate $P(x)$ in Σ_k^0 (in Π_k^0) is called *complete* $\Leftrightarrow$ [for every predicate $S(x)$ in Σ_k^0 (in Π_k^0) there exists a recursive function ψ_S such that $S(x) \Leftrightarrow P(\psi_S(x))$]. Let $\beta = \{x \mid P(x)$ is true$\}$; then β is *complete* (for Σ_k^0 or Π_k^0) $\Leftrightarrow P(x)$ is complete (for Σ_k^0 or Π_k^0, correspondingly). For each $k \geqq 2$, let π_k denote a fixed recursive "k-tupling" function: π_k maps $N \times \cdots \times N$ (k times) one-one onto N. Let $\tau_k^1, \ldots, \tau_k^k$ be the (recursive) "converses" of π_k, i.e.,

$$\pi_k(\tau_k^1(x), \ldots, \tau_k^k(x)) = x,$$

for all x. Thus, for each $k \geqq 2$, $\pi_k(x_1, \ldots, x_k)$ is a "Gödel number" of the ordered k-tuple $\langle x_1, \ldots, x_k\rangle = \langle \tau_k^1(\pi_k(x_1, \ldots, x_k)), \ldots, \tau_k^k(\pi_k(x_1, \ldots, x_k))\rangle$. For each $k \geqq 1$, fix a recursive enumeration $\mathscr{E}_k$ of the set $\{\langle e, x_1, \ldots, x_k, y\rangle \mid \varphi_e^k(x_1, \ldots, x_k) = y\}$; then denote by $\varphi_e^{k,s}$ the set $\{\langle x_1, \ldots, x_k, y\rangle \mid (\exists t)_{t \leqq s}(\mathscr{E}_k(t) = \langle e, x_1, \ldots, x_k, y\rangle)\}$. We denote by p_n the nth prime number in order of magnitude (starting the indexing at $p_0 = 2$) and by P_n the set $\{p_n^r \mid r \in N\}$; then $(m)_n$ denotes, as usual, the power to which p_n divides m (with $(0)_n = 0$ for all n). In addition to r.e. sets, we wish to consider r.e. classes of r.e. sets and also families of r.e. classes of r.e. sets. By an r.e. class we mean a class K of r.e. sets such that, for some e, $K = \{W_x \mid x \in W_e\}$. If $K = \{W_x \mid x \in W_e\}$, we refer to e as an *index* of K; and we define, for each e, $W_e^C = \{W_x \mid x \in W_e\}$. The letters K and L shall henceforth be used to denote classes (whether r.e. or not) of r.e. sets. By an r.e. family we mean a family $\mathscr{F}$ of r.e. classes such that, for some e, $\mathscr{F} = \{W_x^C \mid x \in W_e\}$. If $\mathscr{F} = \{W_x^C \mid x \in W_e\}$, we refer to

e as an *r.e. index* of $\mathscr{F}$; and we define, for each e, $W_e^F = \{W_x^C \mid x \in W_e\}$. More generally, by a Σ_k^0 family ($k \geq 1$) we mean a family $\mathscr{F}$ of r.e. classes such that, for some e, $\mathscr{F} = \{W_x^C \mid x \in \Sigma[k; e]\}$. If $\mathscr{F} = \{W_x^C \mid x \in \Sigma[k; e]\}$, we refer to the pair $\langle k, e \rangle$ as a Σ_k^0 *index* of $\mathscr{F}$; and we define, for each e, $\Sigma[k; e]^F = \{W_x^C \mid x \in \Sigma[k; e]\}$. Capital script letter such as $\mathscr{F}$ and $\mathscr{G}$ are used to denote families (whether r.e. or not) of r.e. classes. If $\mathscr{F}$ is any family of r.e. classes, the *index set*, $G(\mathscr{F})$, corresponding to $\mathscr{F}$ is defined by $G(\mathscr{F}) = \{e \mid W_e^C \in \mathscr{F}\}$. It is easily seen that there exists a recursive function ζ_0 such that for all e we have

$$W_k^C \in W_e^F \Leftrightarrow (\exists j)(\forall h)[W_h \in W_k^C \Leftrightarrow (\exists l)(W_h = \delta\varphi^3_{\zeta_0(e)}(j, l, x))].$$

Conversely, there is a recursive function ζ_1 such that for all e we have

$$W_k^C \in W_{\zeta_1(e)}^F \Leftrightarrow (\exists j)(\forall h)[W_h \in W_k^C \Leftrightarrow (\exists l)(W_h = \delta\varphi^3_e(j, l, x))].$$

Thus, three-place partial recursive functions are effectively interchangeable with collections W_e^F as descriptions of r.e. families. A three-place partial recursive function φ_e^3, regarded as a description of an r.e. family $\mathscr{F}$, shall be termed an *enumeration* of $\mathscr{F}$; and we term φ_e^3 a *row-disjoint enumeration* (of whatever family it enumerates) if the following condition is satisfied:

$$(\forall j)(\forall m)(\forall n)[m \neq n \Rightarrow \delta\varphi_e^3(j, m, x) \cap \delta\varphi_e^3(j, n, x) = \varnothing].$$

It will prove convenient in §3 to work with the class of special retracing functions, i.e., those partial recursive functions which retrace at least one infinite subset of N and possess properties (3) and (4) of [2, p. 81]; we shall here add the requirement that a special retracing function have only finitely many fixed points. Some use will be made of the notation $\gamma^*(x)$ (γ an arbitrary one-place partial number-theoretic function); the meaning of this notation is that prescribed in [2, p. 81]. For some of the basic properties of special retracing functions and of the mapping $\gamma \to \gamma^*$, the reader may consult [1], [2], and [7]. For any partial recursive function γ such that $x \in \delta\gamma \Rightarrow \gamma(x) \leq x$, we denote by K_γ the disjoint r.e. class $\{\{n \mid n \in \delta\gamma \ \&\ \gamma^*(n) = k\} \mid k \in N\}$; and we denote by $\boldsymbol{FR}$ the family $\{K_\gamma \mid \gamma$ is a finite-to-one special retracing function$\}$. Finally, we shall extend the notion of *productive class* (see [3]) in a natural way to the context of *families of r.e. classes*: for a given $k \geq 1$, a family $\mathscr{F}$ of r.e. classes is Σ_k^0-*productive* $\Leftrightarrow$ there exists a partial recursive function ψ such that

$$(\forall e)[\Sigma[k; e]^F \subseteq \mathscr{F} \Rightarrow (e \in \delta\psi \ \&\ W^C_{\psi(e)} \in \mathscr{F} - \Sigma[k; e]^F)];$$

such a function ψ is termed a Σ_k^0-*productive function for* $\mathscr{F}$. If ψ is a Σ_k^0-productive function for $\mathscr{F}$, we denote by $\{\mathscr{F}; k; \psi\}$ the family $\{W^C_{\psi(e)} \mid \Sigma[k; e]^F \subseteq \mathscr{F}\}$ (which in the language of [3] would be called the Σ_k^0-*productive center* of $\mathscr{F}$ with respect to ψ).

2. Classification of index sets corresponding to families.

THEOREM 1. (i) *If* β *is complete at level* Π^0_k, *then* $\{e \mid W_e \subseteq \beta\}$ *is also complete at level* Π^0_k.

(ii) *If* β *is complete at level* Σ^0_k, *then* $\{e \mid W_e \subseteq \beta\}$ *is complete at level* Π^0_{k+1}.

Proof. Routine manipulations (as in [6, §14.3]) show that the predicate $W_x \subseteq \beta$ is in Π^0_k in case (i) and in Π^0_{k+1} in case (ii). Assume β to be Σ^0_k complete, and let $P(x)$ be any Π^0_{k+1} predicate. Express $P(x)$ in the form $(\forall y)Q(x, y)$, where $Q(x, y)$ is a Σ^0_k predicate of x and y. Since β is complete for Σ^0_k predicates, there is a recursive function ψ of two variables such that $(\forall y)Q(x, y) \Leftrightarrow (\forall y)[\psi(x, y) \in \beta]$. So let f be a recursive function such that $(\forall x)[W_{f(x)} = \rho\psi(x, y)]$. Then $P(x) \Leftrightarrow W_{f(x)} \subseteq \beta$, and (ii) is proved. The proof of (i) is similar. ▮

COROLLARY. $G(\text{Fin})$ *is* Π^0_3 *complete.*

Proof. Let $\beta = \{x \mid W_x \text{ is finite}\}$. Then β is Σ^0_2 complete, as is shown in [3]. Hence, by Theorem 1, $\{e \mid W_e \subseteq \beta\}$ is Π^0_3 complete. But obviously $\{e \mid W_e \subseteq \beta\} = G(\text{Fin})$. ▮

The above formulation of Theorem 1 was noted by Carl Jockusch, after its proof had been applied by the author to some special cases.

It is perhaps worth remarking that the index set corresponding to a recursively enumerable family of r.e. classes can reside at any of the following levels of the arithmetical hierarchy: $\Sigma^0_1 \cap \Pi^0_1$ (i.e., recursive), complete Σ^0_1, complete Π^0_1, complete Σ^0_2, complete Π^0_2, complete Σ^0_3, complete Π^0_3, complete Σ^0_4, complete Π^0_4, complete Σ^0_5. Thus the above corollary provides no clue as to the possibility of Σ^0_k-enumerability of Fin for $k \leqq 3$.

3. Σ^0_3-productivity of Fin.

We define Fin* = the family of all disjoint r.e. classes of nonempty finite sets (the empty class not excluded), $\text{Fin}^*_\varnothing$ = the family of all disjoint r.e. classes of finite sets (the empty class not excluded), $\text{Fin}^{*\infty} = \{K \mid K \in \text{Fin}^* \ \& \ K \text{ is infinite}\}$, and $\text{Fin}^{*\infty}_\varnothing = \{K \mid K \in \text{Fin}^*_\varnothing \ \& \ K \text{ is infinite}\}$. It is easy to see that $\boldsymbol{FR} \subseteq \text{Fin}^{*\infty}$ and $\text{Fin}^{*\infty} \nsubseteq \boldsymbol{FR}$; and in fact, by making minor changes in [5, proof of Theorem 3], one can produce much narrower families than $\text{Fin}^{*\infty}$ which properly include $\boldsymbol{FR}$. Our first two lemmas serve as technical lubrication for the proof of Lemma C.

LEMMA A. *There exists a recursive function* ξ *such that*

(1) $W^F_e \subseteq \text{Fin} \Rightarrow \varphi^3_{\xi(e)}$ *is a row-disjoint enumeration of a subfamily of* $\text{Fin}^*_\varnothing$, *and*

(2) $W^C_f \in W^F_e \cap \boldsymbol{FR} \Rightarrow$ *the family enumerated by* $\varphi^3_{\xi(e)}$ *contains the class* W^C_f.

Proof. Let ζ be a recursive function such that $(\forall e)[\varphi^3_{\zeta(e)}$ is an enumeration of $W^F_e]$. We shall apply to each index e a suitable "disjointification" of the rows of the enumeration $\varphi^3_{\zeta(e)}$; our procedure is uniform in e (and results, in general, in some alteration of the given family W^F_e). In order to "disjointify" rows in a way appropriate to proving part (2) of the lemma, we shall treat the functions φ^1_n as candidates for special retracing functions γ whose associated classes K_γ we strive to locate among the rows of the enumeration $\varphi^3_{\zeta(e)}$. In detail, we proceed as follows.

Stage s. Let $(s)_0=a$, $(s)_1=b$, $(s)_2=c$, $(s)_3=d$. We direct our attention to φ_a^1, to $\varphi_{\xi(e)}^3(b, c, x)$, and to the numbers (if any) which φ_a^1 retraces to a fixed point in a minimum of d applications. Specifically, consider $\varphi_{\xi(e)}^{3,s}(b, c, x)$ and $\varphi_a^{1,s}$. If $\varphi_a^{1,s}$ does not have a nonempty subset of the form

$$\{\langle y, \varphi_a^{1,s}(y)\rangle, \langle \varphi_a^{1,s}(y), \varphi_a^{1,s}(\varphi_a^{1,s}(y))\rangle, \ldots, \langle [\varphi_a^{1,s}]^d(y), [\varphi_a^{1,s}]^{d+1}(y)\rangle\}$$

where (i) $y>\varphi_a^{1,s}(y)>\cdots>[\varphi_a^{1,s}]^d(y)$, (ii) $[\varphi_a^{1,s}]^d(y)=[\varphi_a^{1,s}]^{d+1}(y)$, and (iii) $y\in\delta\varphi_{\xi(e)}^{3,s}(b, c, x)$, then we set $\kappa_e^{(s)}\doteq\kappa_e^{(s-1)}$ ($\kappa_e^{(s)}=\varnothing$ if $s=0$) and proceed to stage $s+1$. Suppose, on the other hand, that such a subset of $\varphi_a^{1,s}$ *does* exist. Two cases arise.

Case 1. There is a number $c'\neq c$ such that for some y and some $s'<s$ we have (i′) $\varphi_a^{1,s'}$ contains a subset of the form

$$\{\langle y, \varphi_a^{1,s'}(y)\rangle, \langle \varphi_a^{1,s'}(y), \varphi_a^{1,s'}(\varphi_a^{1,s'}(y))\rangle, \ldots, \langle [\varphi_a^{1,s'}]^d(y), [\varphi_a^{1,s'}]^{d+1}(y)\rangle\},$$

where (ii′) $y>\varphi_a^{1,s'}(y)>\cdots>[\varphi_a^{1,s'}]^d(y)$, (iii′) $[\varphi_a^{1,s'}]^d(y)=[\varphi_a^{1,s'}]^{d+1}(y)$, (iv′) $y\in\delta\varphi_{\xi(e)}^{3,s'}(b, c', x)$, and (v′) $\langle\pi_2(a, b), d, y, 0\rangle\in\kappa_e^{(s')}$.

In this case we set $\kappa_e^{(s)}=\kappa_e^{(s-1)}$ and proceed to stage $s+1$.

Case 2. Case 1 does not hold. Let $x_0, \ldots, x_l$ be all those members m of $\delta\varphi_{\xi(e)}^{3,s}(b, c, x)$ for which there is a subset

$$\{\langle m, \varphi_a^{1,s}(m)\rangle, \langle \varphi_a^{1,s}(m), \varphi_a^{1,s}(\varphi_a^{1,s}(m))\rangle, \ldots, \langle [\varphi_a^{1,s}]^d(m), [\varphi_a^{1,s}]^{d+1}(m)\rangle\}$$

of $\varphi_a^{1,s}$ with $m>\varphi_a^{1,s}(m)>\cdots>[\varphi_a^{1,s}]^d(m)$ and $[\varphi_a^{1,s}]^{d+1}(m)=[\varphi_a^{1,s}]^d(m)$.

Set $\kappa_e^{(s)}=\kappa_e^{(s-1)}\cup\{\langle\pi_2(a, b), d, m, 0\rangle \mid m\in\{x_0, \ldots, x_l\}\}$; then proceed to stage $s+1$.

That completes our description of stage s in the construction of κ_e, where $\kappa_e=\bigcup_s \kappa_e^{(s)}$; we must verify that κ_e is a partial recursive function suitable for use as $\varphi_{\xi(e)}^3$. Certainly κ_e is a function, since the only number which ever occurs as a value $\tau_4^4(n)$, $n\in\kappa_e$, is 0. But it is also plain that κ_e is an r.e. set of quadruples obtained in a uniform effective way from e. Thus, κ_e is a three-place partial recursive function which we may represent as $\varphi_{\xi(e)}^3$, ξ a recursive function of e. We next assume that $W_e^F\subseteq\text{Fin}$, and show that κ_e provides a row-disjoint enumeration of a subfamily of $\text{Fin}_\varnothing^*$. Fix a number n, and consider the nth row enumerated by κ_e. The mth set in this row is given by $\delta\kappa_e(n, m, x)$. Now, it is clear from the construction of κ_e that if $\delta\kappa_e(n, m, x)\neq\varnothing$ then there exists a number c_0 such that $\delta\kappa_e(n, m, x)$ consists entirely of elements of $\delta\varphi_{\xi(e)}^3(\tau_2^2(n), c_0, x)$ which are retraced by $\varphi_{\tau_2^1(n)}^1$ to a fixed point in exactly m steps. But $\delta\varphi_{\xi(e)}^3(\tau_2^2(n), c_0, x)$ is finite since $W_e^F\subseteq\text{Fin}$; hence $\delta\kappa_e(n, m, x)$ is also finite. That κ_e is a *row-disjoint* enumeration of the family which it enumerates is clear from the fact, just cited, that all elements of $\delta\kappa_e(n, m, x)$ have height exactly $=m$ under iterated application of $\varphi_{\tau_2^1(n)}^1$.

It now remains to show that if $W_f^C\in W_e^F\cap \boldsymbol{FR}$, then some row of the enumeration given by κ_e precisely covers the membership of W_f^C. But if $W_f^C\in W_e^F\cap \boldsymbol{FR}$, then there is a special, finite-to-one retracing function φ_a^1 such that $W_f^C = K_{\varphi_a^1}$; moreover, some row $\varphi^3_{\zeta(e)}(b, x, y)$ in the enumeration $\varphi^3_{\zeta(e)}$ of W_e^F enumerates precisely the class W_f^C. It is clear from the construction of κ_e that these two facts imply enumeration of W_f^C by the row $\kappa_e(\pi_2(a, b), x, y)$. ∎

LEMMA B. *There exists a recursive function* μ *such that* $\{\Sigma[3; e]^F \subseteq \text{Fin} \ \& \ x \in \Sigma[3; e]\} \Rightarrow \{\varphi^3_{\mu(x)}$ *is a row-disjoint enumeration of a subfamily* $\mathscr{G}$ *of* $\text{Fin}^*_\varnothing$ *such that* $(W_x^C \in \boldsymbol{FR} \Rightarrow W_x^C$ *is enumerated by some row of the enumeration* $\varphi^3_{\mu(x)})\}$. (*More generally, and in virtue of the same proof, replace* $\Sigma[3; e]$ *by an arbitrary set* α *of natural numbers, and replace* $\Sigma[3; e]^F$ *by the family* $\mathscr{F} = \{W_x^C \mid x \in \alpha\}$.)

Proof. Let ξ be as in Lemma A, and let β be a recursive function such that $W_{\beta(x)} = \{x\}$ holds for all x. Then it is easily deduced from the statement of Lemma A that the function μ defined by $\mu(x) = \xi(\beta(x))$ has the required property. ∎

LEMMA C. *There exists a recursive function* ψ *such that* ψ *is* Σ^0_3*-productive for* Fin *and* $\{\text{Fin}; 3; \psi\} \subseteq \boldsymbol{FR}$.

Proof. We shall make use of a collection $\{\Lambda_{\langle a, b, c\rangle}\}$ of "markers", one for each ordered triple $\langle a, b, c\rangle$ of natural numbers. We impose an ordering on these markers by the rule:

$$\Lambda_{\langle a_1, b_1, c_1\rangle} \prec \Lambda_{\langle a_2, b_2, c_2\rangle} \Leftrightarrow \pi_3(a_1, b_1, c_1) < \pi_3(a_2, b_2, c_2).$$

Let μ be a recursive function as in Lemma B. The rough idea of our procedure is this: We assume a Σ^0_3 predicate $(\exists w)(\forall z)(\exists y)R(x, w, z, y)$ to be fixed, and we consider a particular number a. $\Lambda_{\langle a, b, c\rangle}$ is used to keep track, insofar as possible, of events in the bth row of the enumeration $\varphi^3_{\mu(a)}$; we move $\Lambda_{\langle a, b, c\rangle}$ so as to contribute to the production of a suitable class in $\boldsymbol{FR}$. The movement of $\Lambda_{\langle a, b, c\rangle}$ is restricted by the condition that we may move $\Lambda_{\langle a, b, c\rangle}$ autonomously for the nth time only after having verified that $(\forall z \leqq n)(\exists y)R(a, c, z, y)$; an "autonomous" move of $\Lambda_{\langle a, b, c\rangle}$ is one which is *not* occasioned simply by the fact that some marker $\Lambda_{\langle d, e, f\rangle}$ is moved where $\pi_3(d, e, f) < \pi_3(a, b, c)$. We proceed now to the details of the construction. For notational convenience, we shall abbreviate $\langle \tau_3^1(k), \tau_3^2(k), \tau_3^3(k)\rangle$ to $\langle k\rangle^{-1}$, for all k.

Stage 0. Attach $\Lambda_{\langle 0\rangle^{-1}}$ to 0, set $\alpha^{(0)} = \{\langle 0, 0\rangle\}$, and proceed to stage 1.

Stage s, $s > 0$. We assume it to have been arranged that, at the conclusion of stage $s-1$, exactly the members of the initial segment $\Lambda_{\langle 0\rangle^{-1}}, \Lambda_{\langle 1\rangle^{-1}}, \ldots, \Lambda_{\langle s-1\rangle^{-1}}$ of markers are attached to numbers (that this assumption is allowable will be manifest from our description of the remainder of the construction); and we shall denote by λ_j^{s-1} the number to which $\Lambda_{\langle j\rangle^{-1}}$ is attached at the end of stage $s-1$, $0 \leqq j \leqq s-1$. We shall assume further that a number e has been fixed, and deal throughout the remainder of our description of stage s with the fixed predicate $(\exists w)(\forall z)(\exists y)T_3(e, x, w, z, y)$; once the procedure has been described in full, it will be obvious that it is uniform in e. Now, we wish to move in a suitable way the least marker, if any, whose position is currently *insecure*; so we must explain what we mean in saying that a marker position is currently (i.e., at stage s) insecure. In our definition of insecurity, marker positions will be referred to as being *associated with* certain members of certain rows of certain recursive enumerations of classes; how this association comes about will be clear from the main part of the construction, following the definition. We remind the reader that μ is as in Lemma B.

DEFINITION. Consider λ_j^{s-1}, where $0<j\leqq s-1$; and let $\pi_3^{-1}(j)=\langle a, b, c\rangle$. λ_j^{s-1} is *insecure* if and only if the following conditions are satisfied:

(1) $(\forall z\leqq r+1)(\exists y\leqq s)T_3(e, a, c, z, y)$, where $r=$the number of *previous autonomous moves* (i.e., autonomous moves during stages $t<s$) of $\Lambda_{\langle j\rangle^{-1}}$; and

(2) λ_j^{s-1} is currently associated with a term $\varphi^3_{\mu(a)}(b, t_j, x)$ such that

$$(\exists l \leqq j)(\exists n)[n \in \delta\varphi^{3,s}_{\mu(a)}(b, t_j, x) \ \&\ [(l < j \ \&\ \lambda_l^{s-1} \geqq n) \vee (l = j \ \&\ \lambda_l^{s-1} \leqq n)]].$$

The procedure now splits into cases, according as there does or does not exist an insecure marker position.

Case I. For all j such that $0\leqq j\leqq s-1$, λ_j^{s-1} is not insecure. In this case, our only concern is to attach $\Lambda_{\langle s\rangle^{-1}}$ in a suitable way. Let $\pi_3^{-1}(s)=\langle a_1, b_1, c_1\rangle$. Let t_s be the smallest number t with the following properties:

(i) $(\forall l\leqq s-1)(\forall n \in \delta\varphi^{3,s}_{\mu(a_1)}(b_1, (t)_0, x))(n>\lambda_l^{s-1})$,

(ii) $(\forall n \in \delta\varphi^{3,s}_{\mu(a_1)}(b_1, (t)_0, x))(n<(t)_1)$, and

(iii) $(t)_1>\max\{m \mid (\exists q\leqq s-1)(\exists u\leqq s-1)(m=\lambda_u^q)\}$.

(We remark that the function which maps s to

$$\max\{m \mid (\exists q \leqq s-1)(\exists u \leqq s-1)(m = \lambda_u^q)\}$$

is recursive; this will be evident once our description of stage s is complete.) We attach $\Lambda_{\langle s\rangle^{-1}}$ to $(t_s)_1$, associate $(t_s)_1$ with the term $\varphi^3_{\mu(a_1)}(b_1, (t_s)_0, x)$, set $\alpha^{(s)} =\alpha^{(s-1)} \cup \{\langle (t_s)_1, \lambda_{s-1}^{s-1}\rangle\}$, and proceed to stage $s+1$.

Case II. There exists a number j, $0<j\leqq s-1$, such that λ_j^{s-1} is insecure. Let $j_0=$the least such j, and let $\pi_3^{-1}(j_0)=\langle a_2, b_2, c_2\rangle$. We take t_{j_0} to be the smallest number t with the following properties:

(i′) $(\forall l<j_0)(\forall n \in \delta\varphi^{3,s}_{\mu(a_2)}(b_2, (t)_0, x))(n>\lambda_l^{s-1})$,

(ii′) $(\forall n \in \delta\varphi^{3,s}_{\mu(a_2)}(b_2, (t)_0, x))(n<(t)_1)$, and

(iii′) $(t)_1>\max\{m \mid (\exists q\leqq s-1)(\exists u\leqq s-1)(m=\lambda_u^q)\}$.

If $(\exists l<j_0)(\lambda_l^{s-1}\geqq n$ for some $n \in \delta\varphi^{3,s}_{\mu(a_2)}(b_2, z_0, x))$, where $\varphi^3_{\mu(a_2)}(b_2, z_0, x)$ is the term with which $\lambda_{j_0}^{s-1}$ was associated at the beginning of stage s, then we remove $\Lambda_{\langle j_0\rangle^{-1}}$ from $\lambda_{j_0}^{s-1}$, reattach $\Lambda_{\langle j_0\rangle^{-1}}$ to $(t_{j_0})_1$, associate $(t_{j_0})_1$ with the term $\delta^3_{\mu(a_2)}(b_2, (t_{j_0})_0, x)$, and remove $\Lambda_{\langle k\rangle^{-1}}$ from its present position λ_k^{s-1} for all k such that $j_0<k\leqq s-1$. If no such l exists, then it must be the case that $\lambda_{j_0}^{s-1}\leqq n$ for some $n \in \delta\varphi^{3,s}_{\mu(a_2)}(b_2, z_0, x)$. In this event, we move $\Lambda_{\langle j_0\rangle^{-1}}$ as before, but associate its new position $(t_{j_0})_1$ with $\varphi^3_{\mu(a_2)}(b_2, z_0, x)$ (i.e., with the same term to which $\lambda_{j_0}^{s-1}$ was associated). The movement just imposed on $\Lambda_{\langle j_0\rangle^{-1}}$ counts as an *autonomous move*.

We shall reattach $\Lambda_{\langle j_0+1\rangle^{-1}}, \ldots, \Lambda_{\langle s-1\rangle^{-1}}$ (and, also, shall newly attach $\Lambda_{\langle s\rangle^{-1}}$); but their moves are not counted as autonomous.

Next we reattach $\Lambda_{\langle j_0+1\rangle^{-1}}$ (unless $j_0=s-1$). Let $\pi_3^{-1}(j_0+1)=\langle a_3, b_3, c_3\rangle$; and let t_{j_0+1} be the smallest number t with the following properties:

(i″) $(\forall l<j_0)(\forall\, n\in \delta\varphi^{3,s}_{\mu(a_3)}(b_3, (t)_0, x))(n>\lambda_l^{s-1})$ & $(\forall\, n\in \delta\varphi^{3,s}_{\mu(a_3)}(b_3, (t)_0, x))(n>\lambda_{j_0}^s =(t_{j_0})_1)$,

(ii″) $(\forall n\in \delta\varphi^{3,s}_{\mu(a_3)}(b_3, (t)_0, x))(n<(t)_1)$, and

(iii″) $(t)_1>\max\{m \mid (\exists q\leqq s-1)(\exists u\leqq s-1)[m=\lambda_u^q \vee m=(t_{j_0})_1]\}$.

We attach $\Lambda_{\langle j_0+1\rangle^{-1}}$ to $(t_{j_0+1})_1$ and associate $(t_{j_0+1})_1$ with the term

$$\varphi^3_{\mu(a_3)}(b_3, (t_{j_0+1})_0, x).$$

(Our assumption here is that $j_0<s-1$.) In general, let us suppose that we have accomplished the reattachment of $\Lambda_{\langle j_0+1\rangle^{-1}}, \ldots, \Lambda_{\langle j_0+p\rangle^{-1}}$, $p\geqq 1$, $j_0+p+1<s-1$. Let us designate by $\lambda_0^s, \lambda_1^s, \ldots, \lambda_{j_0}^s, \lambda_{j_0+1}^s, \ldots, \lambda_{j_0+p}^s$ the numbers to which $\Lambda_{\langle 0\rangle^{-1}}, \Lambda_{\langle 1\rangle^{-1}}, \ldots, \Lambda_{\langle j_0\rangle^{-1}}, \Lambda_{\langle j_0+1\rangle^{-1}}, \ldots, \Lambda_{\langle j_0+p\rangle^{-1}}$ are attached following this process of reattachment. Let $\pi_3^{-1}(j_0+p+1)=\langle a_4, b_4, c_4\rangle$; and let t_{j_0+p+1} be the smallest number t with the properties:

(i‴) $(\forall l\leqq j_0+p)(\forall n\in \delta\varphi^{3,s}_{\mu(a_4)}(b_4, (t)_0, x))(n>\lambda_l^s)$,

(ii‴) $(\forall n\in \delta\varphi^{3,s}_{\mu(a_4)}(b_4, (t)_0, x))(n<(t)_1)$, and

(iii‴) $(t)_1>\max\{m \mid (\exists q\leqq s-1)(\exists u\leqq s-1)(m=\lambda_u^q) \vee (\exists u\leqq j_0+p)(m=\lambda_u^s)\}$.

We attach $\Lambda_{\langle j_0+p+1\rangle^{-1}}$ to $(t_{j_0+p+1})_1$ and associate $(t_{j_0+p+1})_1$ with the term $\varphi^3_{\mu(a_4)}(b_4, (t_{j_0+p+1})_0, x)$. We then designate $(t_{j_0+p+1})_1$ as $\lambda_{j_0+p+1}^s$.

This procedure continues until all of $\Lambda_{\langle j_0\rangle^{-1}}, \ldots, \Lambda_{\langle s-1\rangle^{-1}}$ are present in their new positions; it then remains only to attach $\Lambda_{\langle s\rangle^{-1}}$. To attach $\Lambda_{\langle s\rangle^{-1}}$, we proceed (apart from extending α) exactly as in Case I, but of course using the numbers λ_l^s, $l\leqq s-1$, in place of the "old" numbers λ_l^{s-1}. Finally, we make our extension of α as follows:

$$\alpha^{(s)} = \alpha^{(s-1)} \cup \{\langle\lambda_{j_0}^s, \lambda_{j_0-1}^s\rangle, \langle\lambda_{j_0+1}^s, \lambda_{j_0}^s\rangle, \ldots, \langle\lambda_{s-1}^s, \lambda_{s-2}^s\rangle, \langle\lambda_s^s, \lambda_{s-1}^s\rangle\}.$$

We then proceed to stage $s+1$.

That completes our description of stage s.

It remains to see that our construction has the desired effect. We shall first argue that if $\Sigma[3; e]^F\subseteq \text{Fin}$ then, for all k, $\lim_{s\to\infty} \lambda_k^s$ exists; i.e., all markers achieve final positions. Assume, then, that $\Sigma[e; 3]^F\subseteq \text{Fin}$. To begin with, it is obvious from the descriptions given of stages 0 and $s>0$ that $\Lambda_{\langle 0\rangle^{-1}}$, after being attached to 0 at stage 0, never moves; hence $\lambda_0=\lim_{s\to\infty} \lambda_0^s$ exists and $=0$. Now assume that $\lambda_j=\lim_{s\to\infty} \lambda_j^s$ exists for all $j\leqq k$, and consider $\Lambda_{\langle k+1\rangle^{-1}}$. Let t be a stage $\geqq k+1$ (so that λ_{k+1}^u is defined for all $u\geqq t$) such that $\lambda_j^{t-1}=\lambda_j$ for all $j\leqq k$. Thus, $\Lambda_{\langle k+1\rangle^{-1}}$ can only move *autonomously* at stages $u\geqq t$. So it suffices to show that λ_{k+1}^u cannot be insecure at more than finitely many stages $u\geqq t$. Now, if $\lambda_{k+1}^{u_1}$ is insecure at $u_1\geqq t$ (indeed, at *any* stage u_1), then, in particular,

$$(\forall z \leqq r+1)(\exists y \leqq u_1+1)T_3(e, a, c, z, y),$$

where $\langle a, b, c\rangle = \pi_3^{-1}(k+1)$ for some b and where r = the number of previous autonomous moves of $\Lambda_{\langle k+1\rangle^{-1}}$ (note that $r \leqq$ the number of $s < u_1$ for which λ_{k+1}^s is insecure). So, if there are infinitely many $u \geqq t$ for which λ_{k+1}^u is insecure, then, since $\Lambda_{\langle k+1\rangle^{-1}}$ must move autonomously at all such u, we have that $(\forall z)(\exists y)T_3(e, a, c, z, y)$. Thus $(\exists w)(\forall z)(\exists y)T_3(e, a, w, z, y)$; so $a \in \Sigma[3; e]$. Hence, by Lemma B, $\varphi_{\mu(a)}^3$ is a *row-disjoint* enumeration of a family of classes of *finite* sets. By the row-disjointedness of $\varphi_{\mu(a)}^3$, finitely many moves of $\Lambda_{\langle k+1\rangle^{-1}}$ will lead to a position $\lambda_{k+1}^{u_1}$ such that $\lambda_{k+1}^{u_1}$ is associated with a term $\varphi_{\mu(a)}^3(b, t, x)$ for which it is the case that

$$(\forall l)(\forall n)(\forall m)[(l \leqq k \;\&\; n \in \delta\varphi_{\mu(a)}^3(b, m, x) \;\&\; m \geqq t) \Rightarrow \lambda_l < n].$$

But then, having reached such a stage u_1, we see from our description of the general stage of the construction that if $\Lambda_{\langle k+1\rangle^{-1}}$ moves after stage u_1 it does not thereby undergo any further change in the associate of its position. Hence, by the finiteness of the sets in the rows enumerated by $\varphi_{\mu(a)}^3$, some finite number of moves of $\Lambda_{\langle k+1\rangle^{-1}}$ subsequent to stage u_1 will bring it to a position $\lambda_{k+1}^{u_2}$ such that $q \geqq u_2 \Rightarrow \lambda_{k+1}^q$ is not insecure (and, hence, $q \geqq u_2 \Rightarrow \lambda_{k+1}^q = \lim_{s\to\infty}\lambda_{k+1}^s$). So we see that in fact there cannot be an infinite number of distinct values for λ_{k+1}^u; i.e., $\lim_{s\to\infty} \lambda_{k+1}^s$ exists. By induction on q, we thus have that $\lambda_q = \lim_{s\to\infty}\lambda_q^s$ exists for all q.

Next, still under the assumption that $\Sigma[3; e]^F \subseteq \text{Fin}$, we shall argue that the class $K_\alpha = \{\{x \mid \alpha^*(x) \text{ is defined and} = n\} \mid n \in N\}$ is a member of $\boldsymbol{FR}$. It is clear from the construction that $\alpha = \bigcup_s \alpha^{(s)}$ is a partial recursive function with the unique fixed point 0; and it is, moreover, plain from the construction of α that $\rho\alpha \subseteq \delta\alpha$, that $\alpha(n) \leqq n$ for $n \in \delta\alpha$, and that, for each n, $\{x \mid \alpha^*(x) \text{ is defined and} = n\}$ = the set of all numbers of the form λ_n^s, $s = 0, 1, 2, \ldots$. Since, as was proven above, these latter sets are finite, and since they are plainly nonempty and disjoint by the construction, with $\langle \lambda_{n+1}, \lambda_n\rangle \in \alpha$ for all n, we see that α is a finite-to-one special retracing function which in particular retraces the sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ of final marker positions. So $K_\alpha \in \boldsymbol{FR}$.

Now, if $\Sigma[3; e]^F \nsubseteq \text{Fin}$ then we cannot assert that $K_\alpha \in \boldsymbol{FR}$; however, K_α is in every case an r.e. class of disjoint nonempty sets. Moreover, the construction of α is obviously uniform in the sense that an index of K_α is effectively obtainable from e; i.e., there is a recursive function ψ such that, for every e, $W_{\psi(e)}^{\boldsymbol{C}} = K_\alpha$ where α is the function constructed relative to that particular e. As we have just argued, $W_{\psi(e)}^{\boldsymbol{C}}$ is in $\boldsymbol{FR}$ provided $\Sigma[3; e]^F \subseteq \text{Fin}$; so if we can show that $\Sigma[3; e]^F \subseteq \text{Fin} \Rightarrow W_{\psi(e)}^{\boldsymbol{C}} \neq W_a^{\boldsymbol{C}}$ for any a such that $a \in \Sigma[3; e]$, then we will have shown that ψ is a Σ_3^0-productive function for Fin with $\{\text{Fin}; 3; \psi\} \subseteq \boldsymbol{FR}$, as required. Supposing, then, that $\Sigma[3; e]^F \subseteq \text{Fin}$ and that $a \in \Sigma[3; e] \;\&\; W_{\psi(e)}^{\boldsymbol{C}} = W_a^{\boldsymbol{C}}$, we see from Lemma B (using the fact that $W_{\psi(e)}^{\boldsymbol{C}} \in \boldsymbol{FR}$) that $W_{\psi(e)}^{\boldsymbol{C}}$ is (for some b) equal to the class of terms enumerated by the bth row, $\varphi_{\mu(a)}(b, z, x)$, of the row-disjoint enumeration $\varphi_{\mu(a)}^3$ (which we recall enumerates a family $\mathscr{G} \subseteq \text{Fin}_\varnothing^*$). Since $a \in \Sigma[3; e]$, there exists a number c such that $(\forall z)(\exists y)T_3(e, a, c, z, y)$. Therefore, the marker $\Lambda_{\langle m\rangle^{-1}}$, where $\pi_3^{-1}(m)=\langle a, b, c\rangle$, can make as many autonomous moves as are required to insure that its final position, λ_m, has the following property:

$$(\forall l < m)(\forall n)[n \in \delta\varphi^3_{\mu(a)}(b, t(m), x) \Rightarrow \lambda_l < n] \;\&$$
$$(\forall n)[n \in \delta\varphi^3_{\mu(a)}(b, t(m), x) \Rightarrow n < \lambda_m],$$

where $\varphi^3_{\mu(a)}(b, t(m), x)$ is the term with which λ_m is associated. (There is no loss of generality in assuming $m \neq 0$.) Since it is plain from the construction that $q > m \Rightarrow \lambda_q > \lambda_m$, we thus see that the term $\delta\varphi^3_{\mu(a)}(b, t(m), x)$ does not contain any number of the form λ_i, $i=0, 1, 2, \ldots$. But hence $\delta\varphi^3_{\mu(a)}(b, t(m), x)$ is *not* one of the members of $W^C_{\psi(e)}$, and we have a contradiction. This completes the proof of Lemma C. ▮

THEOREM 2. *Let $\mathcal{G}$ be any family such that $\boldsymbol{FR} \subseteq \mathcal{G} \subseteq$ Fin. Then $\mathcal{G}$ is Σ^0_3-productive and hence, in particular, is not Σ^0_3 enumerable.*

Proof. Let ψ be as in Lemma C; then the theorem follows at once, since $\{\text{Fin}; 3; \psi\} \subseteq \boldsymbol{FR}$. ▮

We conclude by remarking that the argument used in proving Theorem 2 seems to be tailored rather snugly to the collection of finite sets and to the class of Σ^0_3 predicates. We lack at present a general procedure wherewith to attack the level-of-enumerability problem for other families. However, by suitably modifying the techniques of this paper, we are able to show that the family Cofin of all r.e. classes of *cofinite* subsets of N is Σ^0_4-productive. (By an application of our Theorem 1 to a theorem of Mostowski and Rogers, G(Cofin) is Π^0_4 complete.) We conjecture, but have no notion how to prove, that the same classification as in the case of Cofin applies to the family of all r.e. classes of *recursive* sets.

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